6125021: Introduction to Combinatorics

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Notes From Last Lecture

• We have made a contract and a whole structure of the class
• We have introduced some basic discrete structures
• We have received many emails from YOU
• Q1: “English majored classes are interesting but hard to understand”
  • A1: I will mark out translations for the key words in the slides.
• Q2: “I’m not a full-time student, what could I do?”
  • A2: It is your choice to take part-time, and it requires you to pay more efforts. This is not a hard course but you cannot pass it if you fail to attend every single time.
Introduction

- **Discrete Structures**
  - Graphs, digraphs(有向图), networks, designs, posets(Partial Ordered Sets, 偏序集), strings, patterns, distributions, coverings, and partitions.

- **Enumeration (枚举)**
  - Permutations(排列), combinations, inclusion/exclusion, generating functions, recurrence relations, and Pólya counting(Polya计数).

- **Algorithms and Optimization**
  - Sorting, spanning trees(生成树), shortest paths, Eulerian circuits(欧拉路径), Hamiltonian cycles(哈密尔顿环), graph coloring, network flows, bipartite matchings, and chain partitions.
Enumeration

• Combinatoric basic is counting
  • Counting the number of distributions of objects into cells

• Example 1: Pirates’ gold problem
  • Five ordered pirates, 100 gold coins, how to distribute?
Example 1: Pirates’ gold problem

• Five ordered pirates, 100 gold coins, how to distribute?
  • \( P = \{p1, p2, p3, p4, p5\} \), assume p5 is the captain (the highest priority)
  • p5 picks plan first, what would he do?

• Reverse thinking

• Lemma 1: the top priority pirate wins more than half of the vote will always survive.
Example 1: Pirates’ gold problem

• Let us generalize the problem
  • We have n pirates, \( P = \{p_1, \ldots, p_n\} \), 100 gold coins, how to distribute?

• Induction method(归纳法)
  • We do not know n, let us assume k, e.g., k = 6,
  \( (P_6, P_5, P_4, P_3, P_2, P_1) \rightarrow (98, 0, 1, 0, 1, 0) \)
Example 1: Pirates’ gold problem

- How about k = 200?

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<td>...</td>
<td>0</td>
<td>1</td>
<td>0</td>
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</tr>
</tbody>
</table>
Example 1: Pirates’ gold problem

• P204:

```
P  P1  P2  P3  P4  ...  P199  P200  P201  P202  P203  P204
P204  Y  N  Y  N  Y  N  N  Y  N  N
```

• P208:

```
P  P1  P2  P3  P4  ...  P200  P201  P202  P203  P204  P205  P206  P207
P208  N  Y  N  Y  Y  Y  Y  Y  Y  N  N  N
```

• Lemma 2: After p201, every two pirates can save their own lives, there exist a series of privates that cannot decide their fates.
  • These pirates can save their lives by supporting the following deciding pirate.
In Combinatorics

• After P201, every pirate that can save his life is \( a(n) \)
  \[ a(0) = 202 \]
  \[ a(n) - a(n-1) + 100 = \left\lfloor a(n)/2 \right\rfloor \]
  That is
  \[ a(n) = 2a(n-1) - 200 \] when \( n \) is even
  \[ a(n) = 2a(n-1) - 199 \] when \( n \) is odd

Solving this, we have \( a(n) = 200 + 2^{n+1} \),
Similarly, we have non-savable pirate = \( 200 + 2^n \)
Results

• The final decision form for any pirate after 201

\[ M = 101 + \begin{cases} 
0 & (n = 0) \\
\frac{4}{3} \left(\frac{\frac{n}{2} - 1}{4^{\frac{n}{2}} - 1}\right) & (n为偶数且n ≠ 0) \\
\frac{2}{3} \left(\frac{\frac{n-1}{2} - 1}{4^{\frac{n-1}{2}} - 1}\right) & (n为奇数)
\end{cases} \]

Equations from [1]

Question: how about we have \( n \) pirates and \( k \) coins?

Example 2: Khalifah’s Legacy

• Khalifah is dying and about to teach the last lesson to his sons

• Three sons, 17 camels, the oldest gets one half, the second gets one third, and last one gets one nineth

• How to divide camels?
Example 2: Khalifah’s Legacy

• We have $S = \{s1, s2, s3\}$, $c = 17$
  • $s1 = c/2$
  • $s2 = c/3$
  • $s3 = c/9$

• Objective: find $s1, s2, s3$

• IEEE format problem
Example 2: Philosophers' beer

• Three philosophers enter a bar
  • Bartender asked, “y’all wanna a beer?”
  • Philosopher A says, “I don’t know”
  • Philosopher B says, “I don’t know”
  • Philosopher C says, “Yes”

• How do you know if they all wanna a beer?
Combinatorics and Graph Theory

• A graph $G$ consists of a vertex set $V$ and a collection $E$ of 2-element subsets of $V$
  • $G = \{V, E\}; \ V = \{v_1, v_2, v_3, \ldots, v_n\}; \ E = \{e_1, e_2, \ldots, e_m\}$
  • $e_1 = (1,2), \ e_2 = (2,3), \ e_3 = (1,1)$

• Example:
Combinatorics and Graph Theory

• Max Cycle = k?
• Max Clique = k? (团)
• Max Dis. = k?
• Max Independent Set = k?
• Short distance from one vertex to another?
Radio Stations?
Combinatorics and Number Theory

• Number theory concerns itself with the properties of the positive integers.

• E.g., number games:

```
  2 6 3
  5 9 9
  6 7 6
  7 8 1
  8 9 ?
```
Number sequencing

• For any positive integer, if it is odd, the next integer is timing three plus one, if it is even, the next number is dividing by two.

• \( \exists \, n > 1, \text{if} \ (n \% 2), \ n = 3n + 1; \text{else,} \ n/ = 2; \)

• Counting the sequence of 24

• Collatz sequences, or the Ulam conjecture (after Stanisław Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem

Question: Is there an integer k, that cannot reach to 1 in Collatz sequencing?
Least Common Multiple (最小公倍数)

• For any number i and j, k%i == k%j == 0, and k is the smallest.

• Algorithm? Factor the number into primes(质数)
  • 351785000 = 2^3 * 3^5 * 4^7 * 19 * 23^2
  • 316752027900 = 2^2 * 3^5 * 5^2 * 7^3 * 11 * 23^4

  • LCM = 2^3 * 3^5 * 4^7 * 7^3 * 11 * 19 * 23^4 = 300,914,426,505,000

• c = 5220070641387698449504000148751379227274095462521
LCM Continued

• $c = 5220070641387698449504000148751379227274095462521$

• $a = 45095080578985454453$ and $b = 115756986668303657898962467957$

• Using matlab, mathematics, or maple to calculate factor($c$), $T($factor($c$)) = 34mins in my laptop and 9s in our cluster
Breaking Time

• G.H. Hardy, A Mathematician’s Apology, Cambridge University Press, p. 140

-“[n]o one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.”

We are using it right now, unfortunately
Combinatorics and Geometry/Image Processing

• There are many problems in geometry that are innately combinatorial
Combinatorics and Geometry/Image Processing

• Each pair of lines intersects

• No point belongs to more than two lines
Combinatorics and Geometry/Image Processing

• We have 882 points on a plane, how many lines can we get? How many regions?

• Given a graph, can we get a non-crossing graph?

• Given a graph with 28323124 vertices and 95645124 edges, can we determine it is a non-crossing graph?
Combinatorics and Optimization

• A few optimization problems that are not continuous, as only integer values for the variables will make sense

• Example: what is the shortest distance from K to C?
Combinatorics and Optimization

- Salesman’s problem
- Traveler's problem extension
- The western world bank robber problem
Combinatorics and Optimization

• Spanning tree problem

• Minimum spanning tree
Take-home Message

• We have discussed a few numeric problems
• We have highlighted a few concepts
• We have discussed the idea of optimization and graph
Project is OUT

Projects
Topic Handout

- Male + female group gets 5% more grades than both female or both male

Process

- One opening report: title, authors, abstract and related work list, Due Sep. 30th.
- One mid report: everything in opening report + design and simulation setup
- One final report: everything as a full technical report
- One final presentation: 10 mins + 5 mins Q&A
Homework Rules

• Email it to ncuhomework@outlook.com!!!!!
• Always submit ON TIME!
• Read the homework handout carefully
• Input the correct subject in your Email writing
  • Email subject

  Course 6125021 Combinatorics Homework 3

• Write your homework using Latex or Markdown, then compile it into PDF or you get 0 pts.
That is it for today

Homework